Fuzzy Sets and Systems 29 (1989) 145–153
North-Holland

A SUBJECTIVE APPROACH FOR RANKING FUZZY NUMBERS

Luis Miguel de CAMPOS IBÁÑEZ and Antonio GONZÁLEZ MUÑOZ
Colegio Universitario de Jaén, Universidad de Granada, 23071 Jaén, Spain

Received February 1987
Revised June 1987

Abstract: Starting from a subjective assignation of weights related to the relative importance of the different level sets, we define a new method of comparison of fuzzy numbers. This procedure is an extension of some well-known indices (Adamo, Tsumura et al., Yager). Some properties of our index are studied in this paper, as well as its behaviour on several particular cases.

Keywords: Fuzzy numbers; crisp order relation; decision-maker's subjectivity.

Introduction

The fuzzy set theory, and particularly the concept of fuzzy number, provides an appropriate theoretical framework to model quantities that are imprecise because of their own nature or some faults in measurement.

Fuzzy numbers have been applied on decision and optimization problems, among others. In these problems, the necessity of procedures to rank fuzzy numbers is obvious.

All the proposed methods can be classified as corresponding to two different approaches:

(A) Ranking fuzzy numbers using crisp relations. This is the case of the procedures based on a ranking function (see Yager [10], Adamo [1] or González [6] as examples).

They lead to a crisp total order relation between fuzzy numbers, induced by the classical order on the real line. Our method could be included in this approach.

(B) To give a comparison index for each pair of fuzzy numbers. In this case, a fuzzy relation is built.

The works of Dubois and Prade [4] and Delgado et al. [3] are examples of this approach.

A good survey of all these methods can be found in Bortolan and Degani [2].

Ranking of fuzzy numbers is a complex problem. Methods in the current literature give no good solution for every problem. For this reason, we have thought about the possibility to define a new ranking index using the decision-maker's subjectivity.

Since results of comparison in real problems affect implicated individuals, their subjectivity should be reflected in the method for ranking.

The index we propose is based on:

(1) a function on selected level sets reflecting the position of the fuzzy numbers on R, and

(2) the integration of the values of this function through weights, which represent the subjective importance of each level set for the decision-maker.

In Section 1 fuzzy numbers and other basic concepts are defined. Section 2 is devoted to defining our procedure of comparison and to study its properties. Moreover this approach is related to previous ones and it is shown in Section 3 how our method generalizes other ones. In Section 4 we make a particular study for triangular fuzzy numbers. Finally, an example is presented in Section 5.

1. Notation and basic definitions

The following definition of fuzzy number is considered:

Definition 1.1. The fuzzy subset $\tilde{A}$ of R (real line) with membership function $A(\cdot)$ is a fuzzy number iff:

(i) $\forall\alpha \in [0, 1]$, $A_\alpha = \{x \in R \mid A(x) \geqslant \alpha\}$ (α-level set of $\tilde{A}$) is a convex set.

(ii) $A(\cdot)$ is an upper semicontinuous function.

(iii) $\tilde{A}$ is normalized, i.e., $\exists m \in R$ such that $A(m) = 1$.

(iv) $\text{supp}(\tilde{A}) = \{x \in R \mid A(x) > 0\}$ is a bounded set of R.

The set $A_1 = \{x \in R \mid A(x) = 1\}$ is called mode of $\tilde{A}$. When $A_1 = \{m\}$, i.e., it is a single point, $\tilde{A}$ is a unimodal fuzzy number.

Dubois and Prade [5] define fuzzy numbers as unimodal fuzzy numbers. We prefer to omit this condition since in our method it is not necessary.

We will denote by N the set of fuzzy numbers.

From Definition 1.1 level sets of a fuzzy number $\tilde{A}$ are closed real intervals

$$A_\alpha = [a_\alpha, b_\alpha] \quad \forall\alpha \in (0, 1].$$

When $\alpha = 0$, we consider A_0 as the closure of $\text{supp}(\tilde{A})$.

Remark. As is well known, a closed real interval I can be considered as a fuzzy number, if we identify it with a fuzzy subset of R whose membership function is equal to the characteristic function of I, that is, if

$$I = [a, b], \quad a, b \in R,$$

then we identify I with the fuzzy subset $\tilde{I}$ whose membership function is

$$I(x) = \begin{cases} 1 & \text{if } x \in [a, b], \\ 0 & \text{if } x \notin [a, b], \end{cases}$$

and it is obvious that $\tilde{I}$ is a fuzzy number.

If $a = b$ then the interval I becomes a real number, so that it can be considered the real line included in the set N, too.

2. Definition and properties of the average index

Following the approach (A), we define a ranking function to compare fuzzy numbers. This function is called 'average index' because it can be interpreted as a weighted average in the following way: First, the decision-maker chooses a subset Y of the unity interval, so that the associated level sets contain the information which is considered outstanding about the imprecise quantity. Next, he assigns a weight, represented by a probability distribution P, to the different elements or measurable subsets of Y. Also, the decision-maker determines a position function giving to each associated level sets a real number. Finally, the index is defined as an average of positions of level sets in Y using P.

So, the basic elements to define the index are:

(1) A set $Y \in \mathcal{P}([0, 1])$.

(2) A probability distribution P on Y.

(3) A function $f_A : Y \to R$ which represents the positions of every level set of $\tilde{A}$ in R.

The most frequent choices for Y are finite and interval sets. Anyway, the three elements of the model can be subjectively determined by the decision-maker.

Definition 2.1. The number

$$V_P(\tilde{A}) = \int_Y f_A(\alpha)\, \mathrm{d}P(\alpha) \quad \forall \tilde{A} \in N$$

is called average index of $\tilde{A}$.

By means of $V_P(\cdot)$ a comparison relation on N is built:

$$\forall \tilde{A}, \tilde{B} \in N \quad \tilde{A} \leqslant \tilde{B} \Leftrightarrow V_P(\tilde{A}) \leqslant V_P(\tilde{B}). \qquad (*)$$

We will say that $\tilde{A}$ is indifferent to $\tilde{B}$ iff their average indices coincide:

$$\tilde{A} \simeq \tilde{B} \Leftrightarrow V_P(\tilde{A}) = V_P(\tilde{B}).$$

Relation $(*)$ is a crisp preorder on N and an order relation on $N/\sim$.

In general, the definition of f_A could be made arbitrarily by the decision-maker. However we propose to choose one point included in each level set A_α of $\tilde{A}$ as value for $f_A(\alpha)$. Thus, we define

$$f_A^\lambda : Y \to R, \quad f_A^\lambda(\alpha) = \lambda b_\alpha + (1 - \lambda) a_\alpha,$$

where $\lambda \in [0, 1]$, $A_\alpha = [a_\alpha, b_\alpha]$, $\tilde{A} \in N$.

The parameter λ is an optimism–pessimism degree, which must be selected by the decision-maker:

When the most advantageous decision is to choose the greatest quantity, an optimistic person would think of the upper extreme of the interval b_α $(\lambda = 1)$, which reflects the greatest possible profit. On the contrary, a pessimistic person would prefer the lower extreme of the interval a_α $(\lambda = 0)$, which represents the least he can win.

When the most advantageous decision is to choose the least quantity, the interpretation is the opposite, with $\lambda = 0$ for optimism and $\lambda = 1$ for pessimism. Thus, if the optimism–pessimism degree of the decision-maker is $\mu \in [0, 1]$, the parameter λ for the function f_A^λ is

$$\lambda = \begin{cases} \mu & \text{if the 'best' is the 'greatest'}, \\ 1-\mu & \text{if the 'best' is the 'least'}. \end{cases}$$

Between the two extreme values $\lambda = 0$ and $\lambda = 1$ there is an attitudes scale for the uncertainty for each decision-maker.

When we use the function f_A^λ we denote the average index of $\tilde{A}$ by $V_P^\lambda(\tilde{A})$.

Now, we study some properties of the average index. Propositions 2.1 and 2.2 study the average index when the fuzzy number is representing a crisp number or a crisp interval, respectively.

Proposition 2.1. *If $a \in R$ then $V_P^\lambda(a) = a$, $\forall \lambda \in [0, 1]$.*

Proof. If $a \in R$ then $f_a^\lambda(\alpha) = a$, $\forall \alpha \in Y$, $\forall \lambda \in [0, 1]$. Therefore $V_P^\lambda(a) = a$. □

Because of this result, the order defined through V_P^λ generalizes the usual order on R.

Proposition 2.2. *Let $I = [a, b]$, $a, b \in R$ and $\lambda \in [0, 1]$. Then*

$$V_P^\lambda(I) = \lambda b + (1-\lambda)a.$$

Proof. $\forall \alpha \in Y$, $I_\alpha = I$. Then

$$V_P^\lambda(I) = \int_Y (\lambda b + (1-\lambda)a)\, \mathrm{d}P(\alpha) = \lambda b + (1-\lambda)a. \quad □$$

The restriction of V_P^λ to crisp sets provides an appropriate method to rank real intervals.

Proposition 2.3. *Let $\tilde{A}$, $\tilde{B} \in N$ and suppose $\oplus$ is the extended addition of fuzzy numbers. Then*

$$V_P^\lambda(\tilde{A} \oplus \tilde{B}) = V_P^\lambda(\tilde{A}) + V_P^\lambda(\tilde{B}) \quad \forall \lambda \in [0, 1].$$

Proof. Using the following result (Nguyen [7]):

$$(\tilde{A} \oplus \tilde{B})_\alpha = A_\alpha + B_\alpha \quad \forall \tilde{A}, \tilde{B} \in N,$$

and putting $A_\alpha = [a_\alpha, b_\alpha]$, $B_\alpha = [c_\alpha, d_\alpha]$, it follows that

$$(\tilde{A} \oplus \tilde{B})_\alpha = [a_\alpha + c_\alpha, b_\alpha + d_\alpha].$$

Therefore

$$f_{A \oplus B}^\lambda(\alpha) = f_A^\lambda(\alpha) + f_B^\lambda(\alpha) \quad \forall \alpha \in Y,$$

and the result is obtained because of the linearity of the integral operator. □

Proposition 2.4. *Let $r \in R$ and $\tilde{A} \in N$. Then*

$$V_P^\lambda(\widetilde{rA}) = rV_P^\lambda(\tilde{A}) \quad \forall \lambda \in [0, 1].$$

Proof. As $(\widetilde{rA})_\alpha = rA_\alpha \; \forall \alpha \in Y$, it follows that

$$f_{rA}^\lambda(\alpha) = rf_A^\lambda(\alpha) \quad \forall \alpha \in Y$$

and again because of the linearity of the integral operator the result holds. □

Proposition 2.5. *Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D} \in N$. If $\tilde{A} \leqslant \tilde{B}$ and $\tilde{C} \leqslant \tilde{D}$ then*

$$\tilde{A} \oplus \tilde{C} \leqslant \tilde{B} \oplus \tilde{D}.$$

Proof. The result follows from Proposition 2.3. □

Proposition 2.6. *Let $\tilde{A} \in N$ be a unimodal fuzzy number with symmetric membership function around the mode m of $\tilde{A}$. Then*

$$V_P^{1/2}(\tilde{A}) = m.$$

Proof. Since the membership function of $\tilde{A}$ is symmetric around m,

$$A_\alpha = [m - c_\alpha, m + c_\alpha],$$

whence we conclude that $f_A^{1/2}(\alpha) = m \; \forall \alpha \in Y$. Therefore $V_P^{1/2}(\tilde{A}) = m$. □

Comparison between unimodal fuzzy numbers with symmetric membership function using $V_P^{1/2}$ is equivalent to the comparison of their modal values.

3. The average index as extension of some known indices

In this section we will see that the average index coincides with other comparison indices when we use particular sets Y and probability distributions.

Example 3.1. Let $Y = [0, 1]$ and

$$P_0(\alpha) = \begin{cases} 1 & \alpha = \alpha_0, \\ 0 & \text{otherwise}, \end{cases} \quad \alpha_0 \in Y.$$

For $\lambda = 1$ we have

$$\forall \tilde{A} \in N \quad V_{P_0}^1(\tilde{A}) = \sup\{x \mid A(x) \geqslant \alpha_0\},$$

that is the index defined by Adamo [1].

Example 3.2. Let $Y = [0, 1]$ and

$$P_T((a, b]) = b^2 - a^2, \quad (a, b] \in \mathcal{P}([0, 1]).$$

Then, for $\lambda = \frac{1}{2}$ the average index is

$$V_P^{1/2}(\tilde{A}) = \int_0^1 (b_\alpha + a_\alpha)\tfrac{1}{2}\,\mathrm{d}P_T(\alpha) = \int_0^1 (b_\alpha + a_\alpha)\alpha\,\mathrm{d}\alpha \quad \forall \tilde{A} \in N,$$

which coincides with the index proposed by Tsumura et al. [8].

Example 3.3. Let $Y = [0, 1]$ and let P_L be the Lebesgue measure on [0, 1]

$$P_L((a, b]) = b - a, \quad (a, b] \in \mathcal{P}([0, 1]).$$

For $\lambda = \frac{1}{2}$ we have

$$\forall \tilde{A} \in N \quad V_P^{1/2}(\tilde{A}) = \int_0^1 (b_\alpha + a_\alpha)\tfrac{1}{2}\,\mathrm{d}P_L(\alpha) = \int_0^{\alpha_{max}} (b_\alpha + a_\alpha)\tfrac{1}{2}\,\mathrm{d}\alpha$$

where $\alpha_{max} = \sup_{x\in R} A(x)$. This is the index defined by Yager [10].

4. The average index for triangular fuzzy numbers

Let T be the set of fuzzy numbers with triangular membership function (see Figure 1) characterized by four parameters m_1, m_2, a and b. We will denote elements of T by $\tilde{A} = (m_1, m_2, a, b)$.

For triangular fuzzy numbers, the function f_A^λ is

$$f_A^\lambda(\alpha) = m_\lambda + c_\lambda(1 - \alpha) \quad \forall \alpha \in [0, 1],$$

where $m_\lambda = \lambda m_2 + (1 - \lambda)m_1$ and $c_\lambda = \lambda b - (1 - \lambda)a$.

We use as probability measure a normalized Stieltjes measure S on $Y = [0, 1]$,

$$S((a, b]) = s(b) - s(a), \quad (a, b) \in \mathcal{P}([0, 1]),$$

defined through a function $s: [0, 1] \to [0, 1]$ satisfying

(i) $s(0) = 0$, $s(1) = 1$;

(ii) s is a monotone increasing function.

The average index, in this case, has the form

$$V_S^\lambda(\tilde{A}) = \int_0^1 f_A^\lambda(\alpha)\,\mathrm{d}S(\alpha) = \int_0^1 (m_\lambda + c_\lambda(1 - \alpha))\,\mathrm{d}S(\alpha)$$

$$= m_\lambda + c_\lambda\left(1 - \int_0^1 \alpha\,\mathrm{d}S(\alpha)\right).$$

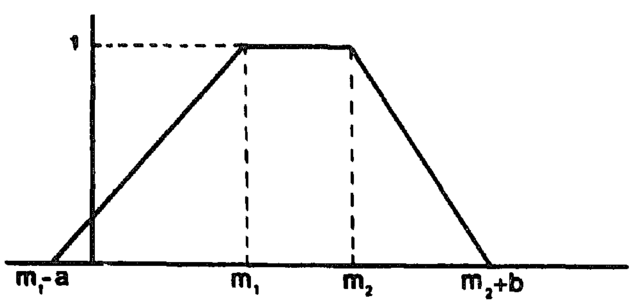

Fig. 1. General form of elements of T.

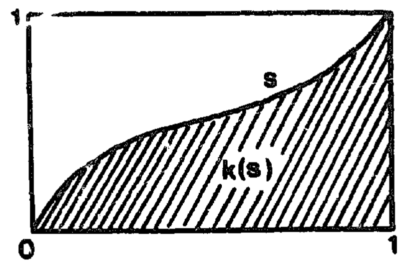

Fig. 2. $k(s)$ is the area below the function s.

Using the well-known equality

$$\int_Y h(x)\,\mathrm{d}S(x) = \int_0^{+\infty} S(\{x \mid h(x) \geq \alpha\})\,\mathrm{d}\alpha$$

we obtain

$$\int_Y \alpha\,\mathrm{d}S(\alpha) = \int_0^{+\infty} S(\{x \mid x \geq \alpha\})\,\mathrm{d}\alpha$$
$$= \int_0^1 S([\alpha, 1])\,\mathrm{d}\alpha = \int_0^1 (s(1) - s(\alpha))\,\mathrm{d}\alpha = 1 - \int_0^1 s(\alpha)\,\mathrm{d}\alpha.$$

If we denote

$$k(s) = \int_0^1 s(\alpha)\,\mathrm{d}\alpha,$$

which is a value depending only on the selected Stieltjes measure (see Figure 2), then

$$V_s^\lambda(A) = m_\lambda + c_\lambda k(s).$$

It is obvious that the average index is

$$V_s^\lambda(\tilde{A}) = f_{\tilde{A}}^\lambda(1 - k(s)).$$

Thus, given a measure, S, $\alpha_s = 1 - k(s)$ is called level of comparison, because the comparison relation is made through the value of $f_{\tilde{A}}^\lambda$ in such level (see Figure 3).

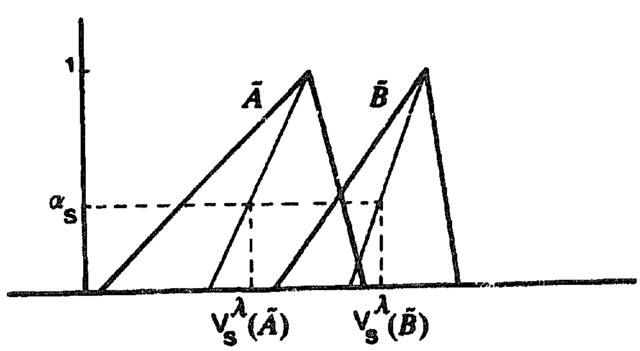

Fig. 3. Comparison between triangular fuzzy numbers using a Stieltjes measure.

When S is the Lebesgue measure ($s(x) = x$), then

$$\alpha_L = \tfrac{1}{2} \quad \text{and} \quad V_L^\lambda(\tilde{A}) = f_A^\lambda(\tfrac{1}{2}) = m_\lambda + \tfrac{1}{2}c_\lambda.$$

So, a Stieltjes measure more general than Lebesgue measure represents only a modification in the level of comparison. In addition any other Stieltjes measure with $\alpha_s = \frac{1}{2}$ gives the same order relation as that given by the Lebesgue measure.

5. An example of fuzzy numbers comparison

We consider fuzzy numbers $\tilde{A}$ and $\tilde{B}$ as in Figure 4, and a decision-maker with optimism–pessimism degree $\lambda \in [0, 1]$. If we use on $Y = [0, 1]$ the Lebesgue measure, average indices are

$$V_L^\lambda(\tilde{A}) = \tfrac{3}{2} + \lambda, \qquad V_L^\lambda(B) = 1 + 2\lambda.$$

Then

$$\tilde{A} \leqslant \tilde{B} \Leftrightarrow V_L^\lambda(\tilde{A}) \leqslant V_L^\lambda(\tilde{B}) \Leftrightarrow \tfrac{3}{2} + \lambda \leqslant 1 + 2\lambda \Leftrightarrow \lambda \geqslant \tfrac{1}{2}.$$

Therefore

$$\tilde{A} \leqslant \tilde{B} \Leftrightarrow \lambda \geqslant \tfrac{1}{2}, \qquad \tilde{B} \leqslant \tilde{A} \Leftrightarrow \lambda \leqslant \tfrac{1}{2}, \qquad \tilde{A} \simeq \tilde{B} \Leftrightarrow \lambda = \tfrac{1}{2}.$$

When $\lambda > \frac{1}{2}$ the decision-maker is more optimistic than usual and therefore he prefers $\tilde{B}$ because it permits him a greatest possible profit. On the other hand ($\lambda < \frac{1}{2}$), a decision-maker more pessimistic than usual prefers A, which assures him a greatest minimum profit. For a decision-maker without optimism–pessimism bias ($\lambda = \frac{1}{2}$), both quantities are indifferent.

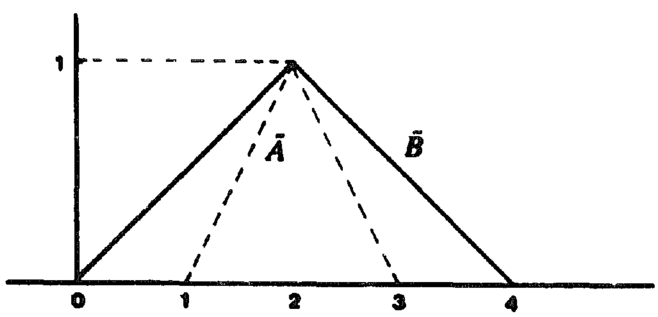

Fig. 4. Fuzzy numbers $\tilde{A} = (2, 1, 1)$ and $\tilde{B} = (2, 2, 2)$.

6. Conclusions

We have defined a method to rank fuzzy numbers that allows us to use the decision-maker's subjectivity. It has good properties and it generalizes some well-known procedures.

However, the nature of the necessary elements to calculate the index $V_P^\lambda(\tilde{A})$ makes its implementation difficult. So, for future work, it seems interesting for us to investigate this problem, studying the different ways to fix P and Y in the framework of each particular problem.

Acknowledgement

The authors are grateful to one of the anonymous referees for valuable comments and suggestions.

References

[1] J.M. Adamo, Fuzzy decision trees, *Fuzzy Sets and Systems* **4** (1980) 207–219.
[2] G. Bortolan and P. Degani, A review of some methods for ranking fuzzy subsets, *Fuzzy Sets and Systems* **15** (1985) 1–19.
[3] M. Delgado, J.L. Verdegay and M.A. Vila, A procedure for ranking fuzzy numbers using fuzzy relations, *Fuzzy Sets and Systems* **26** (1988) 49–62.
[4] D. Dubois and H. Prade, Ranking fuzzy numbers in the setting of possibility theory, *Inform. Sci.* **30** (1983) 183–224.
[5] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications* (Academic Press, New York, 1980).
[6] A. González, Comparaciones entre números difusos por niveles de importancia subjetiva, *Actas de la XV Reunion Nacional de la S.E.I.O.* (1985).
[7] H.T. Nguyen, A note on the extension principle for fuzzy sets, *J. Math. Anal. Appl.* **64** (1978) 369–380.
[8] Y. Tsumura, T. Terano and M. Sugeno, Fuzzy fault tree analysis, *Summary of Papers on General Fuzzy problems, Report* no. 7 (1981) 21–25.
[9] R.R. Yager, Ranking fuzzy subsets over the unit interval, *Proc. 1978 CDC* (1978) 1435–1437.
[10] R.R. Yager, A procedure for ordering fuzzy subsets of the unit interval, *Inform. Sci.* **24** (1981) 143–161.