Fuzzy Sets and Systems 31 (1989) 23–36
North-Holland

REPRESENTATION OF FUZZY MEASURES THROUGH PROBABILITIES

Luis Miguel de CAMPOS IBAÑEZ

Departamento de Ciencias de la Computación e Inteligencia Artificial, Colegio Universitario de Jaén, Universidad de Granada, 23071 Jaén, Spain

Manuel Jorge BOLAÑOS CARMONA

Facultad de Medicina, Universidad de Granada, 18071 Granada, Spain

Received August 1987
Revised December 1987

Abstract: In this paper we develop a method to study fuzzy measures associating certain sets of probabilities to them. This development is based on the consideration of one property of a fuzzy integral, called monotone expectation. Some results about the monotone expectation in relation to associated probabilities are also obtained.

Keywords: Fuzzy measure; monotone expectation; associated probabilities; capacity of order two.

1. Introduction

Additivity does not seem suitable as a demandable property of set functions in many real situations, due to the lack of additivity in many facets of human reasoning. It is sometimes more appropriate to consider non-additive but monotone valuations to express human subjectivity. For example, consider an illness y such that if two symptoms x_1 and x_2 appear simultaneously, then a doctor diagnoses such illness with total certainty, but the appearance of only one symptom is scarcely a sign of y. The belief of a doctor about when a patient has or has not y (taking into account the symptoms x_1 and x_2) may be represented by means of a fuzzy measure better than by an additive measure. (y can be 'migraine headache', x_1 = 'general discomfort' and x_2 = 'pain in the temples, eyes and forehead').

Since Sugeno's definition of fuzzy measure [9], numerous works on the matter have been done. As the field of general fuzzy measures is extremely wide, their study has been frequently tackled by means of particular classes of them.

Nevertheless, a systematization of the general class of fuzzy measures is necessary. The expression of fuzzy measures in terms of probabilities has been one of the richest mechanisms to analyse some particular classes; for instance, it explains the development of Dempster–Shafer's Theory of evidence [4, 8].

The concept of a fuzzy integral appears simultaneously with the fuzzy measure concept. It is either a tool able to condense the information provided by a function in a single value, in terms of a underlying fuzzy measure, or a method to extend fuzzy measures defined on crisp subsets of a referential to fuzzy subsets.

The purpose of this paper is to develop a general methodology to study fuzzy measures associating certain sets of probabilities to them. This development is based on the consideration of one property of a fuzzy integral, called monotone expectation.

In Section 2, definitions of fuzzy measure, monotone expectation and its more important properties are presented. We also make a brief study of the monotone expectation, which justifies our definition of associated probabilities. Section 3 is devoted to the study of such associated probabilities, and, in Section 4, some results about the monotone expectation in relation to associated probabilities are obtained.

2. Basic definitions. Probabilities associated to a fuzzy measure

We start with the definition of fuzzy measure (Sugeno [9]), adapted to the case of a finite referential, which is the only one considered in this work.

Definition 2.1. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and g a set function

$$g : \mathcal{P}(X) \to [0, 1]$$

where $\mathcal{P}(X)$ is the power set of X. We will say g is a fuzzy measure on X if it satisfies:

(i) $g(\emptyset) = 0$; $g(X) = 1$.

(ii) $\forall A, B \subseteq X$, if $A \subseteq B$ then $g(A) \leq g(B)$.

A fuzzy measure is a normalized and monotone set function. It can be considered as an extension of the probability concept, where additivity is replaced by the weaker condition of monotonicity.

Duality is an important concept to study fuzzy measures:

Definition 2.2. Let g be a fuzzy measure on X. The fuzzy measure g^* is defined by

$$g^*(A) = 1 - g(\bar{A}) \quad \forall A \subseteq X,$$

where $\bar{A}$ is the complement of A; g^* is called the dual fuzzy measure of g.

As g is obviously the dual fuzzy measure of g^* too, we will call them dual fuzzy measures, and denote them by (g, g^*).

The concept of duality is very important, since it permits one to obtain alternative representations of a piece of information. So, we will consider a fuzzy measure and its dual measure to contain the same information, but codified in a different way.

A specially interesting class of fuzzy measures are the capacities of order two (Choquet [3]), because it covers a great number of fuzzy measures, and at the same time, capacities of order two possess enough mathematical properties.

Therefore, this class of fuzzy measures combines generality and operativity. As such fuzzy measures will be considered later, we present their definition now:

Definition 2.3. Let (g, g^*) be a pair of dual fuzzy measures.
g is a lower capacity of order two if and only if

$$\forall A, B \subseteq X, \quad g(A \cup B) + g(A \cap B) \geq g(A) + g(B).$$

g^* is an upper capacity of order two if and only if

$$\forall A, B \subseteq X, \quad g^*(A \cup B) + g^*(A \cap B) \leq g^*(A) + g^*(B).$$

The most used classes of fuzzy measures, as belief and plausibility measures (Shafer [8]), necessity and possibility ones (Zadeh [11]), λ-measures (Sugeno [9]) and probabilities, are capacities of order two.

Example 2.1. Consider the three typical symptoms x_1, x_2, x_3 of an illness y, and the following information: 80% of the patients having y present symptoms x_1 and x_2, and 20% have symptoms x_1 and x_3.

This information can be represented by the pair of dual capacities of order two (g, g^*), shown in Table 1.

Table 1

A	$g(A)$	$g^*(A)$
$\{x_1\}$	0	1
$\{x_2\}$	0	0.8
$\{x_3\}$	0	0.2
$\{x_1, x_2\}$	0.8	1
$\{x_1, x_3\}$	0.2	1
$\{x_2, x_3\}$	0	1
$\{x_1, x_2, x_3\}$	1	1

The values $g(A)$ and $g^*(A)$ represent respectively the minimum and maximum degrees of belief (in the light of the given information) about the fact that a patient with the set of symptoms A has the illness y.

A fuzzy integral is a functional that, a fuzzy measure having been fixed, assigns a real value to each function, which represents the average of the function in terms of the given measure. Several types of fuzzy integrals, as the ones defined by Sugeno [9], Kruse [6], Weber [10], ... can be found in the current literature. The following fuzzy integral (based on the Choquet operator [3]) is the monotone expectation, which was defined by Bolaños et al. [1]:

Definition 2.4. Let g be a fuzzy measure on X and $h: X \to \mathbb{R}_0^+$ a non-negative function. The monotone expectation of h with respect to g is

$$E_g(h) = \int_0^{+\infty} g(H_\alpha)\, d\alpha,$$

where $H_\alpha = \{x \in X \mid h(x) \geq \alpha\}$.

The monotone expectation always exists and it is finite for each g and h. It is obvious that $E_g(\cdot)$ is a generalization of the mathematical expectation: that is what it becomes when the used fuzzy measure is a probability, that is,

$$E_P(h) = \int_X h \, dP,$$

if P is a probability measure.

Some of the most important properties of the monotone expectation are:

(1) If $h(x) \leq h'(x) \, \forall x \in X$, then $E_g(h) \leq E_g(h')$.

(2) If $g(A) \leq g'(A) \, \forall A \subseteq X$, then $E_g(h) \leq E_{g'}(h)$.

(3) $E_g(I_A) = g(A)$, if I_A is the characteristic function of $A \subseteq X$.

(4) If $c, b \in \mathbb{R}_0^+$, then $E_g(c + bh) = c + bE_g(h)$.

(5) The monotone expectation is an additive functional only when the fuzzy measure g is a probability.

Proofs of these properties can be found in Bolaños et al. [1]. (For a more detailed study about the monotone expectation, see de Campos [2].)

Remark. We require a non-negative $h(x)$ because the equality

$$\int_X h \, dP = \int_0^{+\infty} P([h \geq x]) \, dx, \quad P \text{ is a probability},$$

is only true for non-negative functions. Nevertheless it is possible to define E_g for any function, but if we wish that E_g preserve some properties of the mathematical expectation (e.g. monotonicity), we must define E_g (see Lamata [7]) as

$$E_g(h) = E_g(h^+) - E_{g*}(h^-),$$

where $h^+(x) = \max(h(x), 0)$ and $h^-(x) = \max(-h(x), 0)$.

Since the monotone expectation is a generalization of the mathematical expectation, it can be questioned whether the former possesses some weaker property in relation to additivity than the latter. The following proposition gives an expression of the monotone expectation that permits us to analyse that question.

Proposition 2.1. *If the values of a non-negative function h are ordered as*

$$h(x_1) \leq h(x_2) \leq \cdots \leq h(x_n),$$

then the monotone expectation of h with respect to a fuzzy measure g can be written as

$$E_g(h) = \sum_{i=1}^{n-1} h(x_i)(g(A_i) - g(A_{i+1})) + h(x_n)g(A_n), \tag{1}$$

where $A_i = \{x_i, x_{i+1}, \ldots, x_n\}$, $i = 1, \ldots, n$.

Proof. The sets H_α are

$$H_\alpha = \begin{cases} \emptyset & \text{if } \alpha > h(x_n), \\ \{x_i, x_{i+1}, \ldots, x_n\} = A_i & \text{if } h(x_{i-1}) < \alpha \leq h(x_i),\ i = 2, \ldots, n, \\ \{x_1, x_2, \ldots, x_n\} = A_1 = X & \text{if } \alpha \leq h(x_1). \end{cases}$$

Therefore

$$\begin{aligned} E_g(h) &= \int_0^{h(x_1)} g(H_\alpha)\,\mathrm{d}\alpha + \sum_{i=2}^{n} \int_{h(x_{i-1})}^{h(x_i)} g(H_\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{h(x_n)}^{+\infty} g(H_\alpha)\,\mathrm{d}\alpha \\ &= h(x_1)g(X) + \sum_{i=2}^{n} (h(x_i) - h(x_{i-1}))g(A_i). \end{aligned}$$

If we develop and regroup this expression, we have

$$E_g(h) = \sum_{i=1}^{n-1} h(x_i)(g(A_i) - g(A_{i+1})) + h(x_n)g(A_n). \qquad \square$$

Expression (1) permits us to consider the following matter: two functions with the same ordering in their values have equal sets A_i (whose form depends only on this order). So, the monotone expectation of their sum is the sum of both monotone expectations, because the function sum ranks its values in the same way. Thus, the monotone expectation is an additive functional for functions ordered equally.

We can also notice that $E_g(h)$ is an average of the h function values weighed by

$$p_i = g(A_i) - g(A_{i+1}), \quad i = 1, \ldots, n-1, \qquad p_n = g(A_n).$$

As

$$\sum_{i=1}^{n} p_i = g(A_1) = g(X) = 1 \quad \text{and} \quad p_i \geq 0,\ i = 1, \ldots, n,$$

the values p_i can be interpreted as the values of a probability function. Then $E_g(h)$ is equivalent to the mathematical expectation of h with respect to that probability distribution.

The values p_i depend on the fuzzy measure g and the sets A_i, which depend on h only in the order determined by its values. So, we can say:

Proposition 2.2. *The monotone expectation of a non-negative function h with respect to a fuzzy measure g coincides with the mathematical expectation of h with respect to a probability that depends only on g and the ordering of the values of h.*

Moreover, the maximum number of probability distributions we need to integrate any function, coincides with the number of possible orderings or permutations in a set with n elements, that is, $n!$.

Thus, it makes sense to associate the $n!$ probabilities to each fuzzy measure, provided that they are deduced from this fuzzy measure through the different possible orderings.

For example, the associated probability for the ordering $(x_n, x_{n-1}, \ldots, x_2, x_1)$ is

$$p_{(n,n-1,\ldots,2,1)}(x_1) = 1 - g(\{x_2, \ldots, x_n\}),$$
$$p_{(n,n-1,\ldots,2,1)}(x_2) = g(\{x_2, x_3, \ldots, x_n\}) - g(\{x_3, \ldots, x_n\}), \ \ldots,$$
$$p_{(n,n-1,\ldots,2,1)}(x_{n-1}) = g(\{x_{n-1}, x_n\}) - g(\{x_n\}),$$
$$p_{(n,n-1,\ldots,2,1)}(x_n) = g(\{x_n\}).$$

In general, the possible orderings of the elements of X are given by the permutations of a set with n elements, which form the group S_n.

Definition 2.5. The probability functions P_σ defined by

$$p_\sigma(x_{\sigma(1)}) = g(\{x_{\sigma(1)}\}), \ \ldots,$$
$$p_\sigma(x_{\sigma(i)}) = g(\{x_{\sigma(1)}, \ldots, x_{\sigma(i)}\}) - g(\{x_{\sigma(1)}, \ldots, x_{\sigma(i-1)}\}), \ \ldots,$$
$$p_\sigma(x_{\sigma(n)}) = 1 - g(\{x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}\}),$$

for each $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in S_n$, are called the associated probabilities of the fuzzy measure g.

So, we have a single ordered set of $n!$ probability measures for each fuzzy measure.

3. Properties of the probabilities associated to a fuzzy measure

In general, knowledge of the associated probabilities to a fuzzy measure g does not permit one to rebuild the measure g without also knowing what permutation generates each one of them. This happens because different fuzzy measures can generate the same set of associated probabilities, as is seen in the following example:

Example 3.1. Consider the fuzzy measures g and g' on $X = \{x_1, x_2\}$ defined in Table 2.

Table 2

A	$g(A)$	$g'(A)$
$\{x_1\}$	0	0.5
$\{x_2\}$	0.5	1
$\{x_1, x_2\}$	1	1

The associated probabilities to g are:
(a) For the permutation $\sigma = (1, 2)$, $p_{(1,2)}(x_1) = 0$, $p_{(1,2)}(x_2) = 1$.
(b) For $\sigma = (2, 1)$, $p_{(2,1)}(x_1) = 0.5$, $p_{(2,1)}(x_2) = 0.5$.
The associated probabilities to g' are:
(a) For $\sigma = (1, 2)$, $p'_{(1,2)}(x_1) = 0.5$, $p'_{(1,2)}(x_2) = 0.5$.
(b) For $\sigma = (2, 1)$, $p'_{(2,1)}(x_1) = 0$, $p'_{(2,1)}(x_2) = 1$.

Therefore g and g' have the same associated probabilities, although they are generated by different permutations. This happens because g' is the dual fuzzy measure of g ($g' = g^*$), as we will see in Proposition 3.2.

The fuzzy measure g can illustrate the example given in the introduction about the migraine: general discomfort (considered by oneself) is nothing representative for this illness; pain in the temples, eyes and forehead is moderately representative, but the two symptoms together are decisive to diagnose migraine.

As in Example 2.1, $g(A)$ and $g^*(A)$ represent respectively the minimum and maximum degrees of belief of a doctor for diagnosing migraine, when the patient has the set of symptoms A.

The associated probabilities are the extreme points of the set of probabilities lying between g and g^*.

The monotone expectation has the following usefulness in this case: symptoms can be partially verified (e.g. the pain may be 'intense', 'moderate',...). If we assign a number in the unity interval representing the degree in which each symptom is verified, the monotone expectation with respect to g or g^* of the function constructed in this way, again represents belief about the illness for a patient having that fuzzy set of symptoms.

The order fulfils a decisive role in order to determine the fuzzy measure starting from its associated probabilities:

Proposition 3.1. *The set of associated probabilities to a fuzzy measure determines the latter if the permutations corresponding to each probability are known.*

Proof. It suffices to prove that if two fuzzy measures g and g' have the same set of associated probabilities, and the latter correspond to the same permutations, then both measures coincide, that is, $P_\sigma = P'_\sigma \forall \sigma \in S_n \Rightarrow g = g'$.

Consider any set $A \subseteq X$. There is always a permutation that puts elements of A in the first places, that is, if

$$A = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\},$$

there exists $\sigma_0 \in S_n$ such that

$$\sigma_0(1) = i_1, \ \ldots, \ \sigma_0(k) = i_k.$$

Under these conditions,

$$\begin{aligned}g(A) &= g(\{x_{i_1},\dots,x_{i_k}\})\\ &= g(\{x_{i_1},\dots,x_{i_k}\}) - g(\{x_{i_1},\dots,x_{i_{k-1}}\})\\ &\quad + g(\{x_{i_1},\dots,x_{i_{k-1}}\}) - \cdots - g(\{x_{i_1},x_{i_2}\})\\ &\quad + g(\{x_{i_1},x_{i_2}\}) - g(\{x_{i_1}\}) + g(\{x_{i_1}\})\\ &= p_{\sigma_0}(x_{i_k}) + \cdots + p_{\sigma_0}(x_{i_1}) = \sum_{j=1}^{k} p_{\sigma_0}(x_{i_j}) = P_{\sigma_0}(A).\end{aligned}$$

We obtain $g'(A) = P'_{\sigma_0}(A)$ similarly.
Therefore, from $P_{\sigma_0}(A) = P'_{\sigma_0}(A)$, $g(A) = g'(A)$ is deduced. □

This proposition permits us to interpret any fuzzy measure in terms of its associated probabilities (which can be equal among each other). In this way, as we have seen, the ordering is basic, and we cannot omit it, although later we will prove we can do it with the wide class of capacities of order two.

The most remarkable case where different fuzzy measures provide the same $n!$ probabilities, but ordered in a different way, is the case of dual fuzzy measures. Before exposing it in the following proposition, we need a definition:

Definition 3.1. We will say that two permutations σ, $\sigma^* \in S_n$ are dual if

$$\sigma^*(i) = \sigma(n-i+1), \quad i = 1, \dots, n.$$

Proposition 3.2. *A necessary and sufficient condition for two fuzzy measures g and g^* to be dual is to have the same $n!$ associated probabilities corresponding to dual permutations, that is, $P_\sigma = P^*_{\sigma^*}$ if σ and σ^* are dual.*

Proof. *Necessary condition*: Consider two dual permutations σ and σ^*:

$$\begin{aligned}p^*_{\sigma^*}(x_{\sigma^*(i)}) &= g^*(\{x_{\sigma^*(1)},\dots,x_{\sigma^*(i)}\}) - g^*(\{x_{\sigma^*(1)},\dots,x_{\sigma^*(i-1)}\})\\ &= 1 - g(\{x_{\sigma^*(n)},\dots,x_{\sigma^*(i+1)}\}) - 1 + g(\{x_{\sigma^*(n)},\dots,x_{\sigma^*(i)}\})\\ &= -g(\{x_{\sigma(1)},\dots,x_{\sigma(n-i)}\}) + g(\{x_{\sigma(1)},\dots,x_{\sigma(n-i+1)}\})\\ &= p_\sigma(x_{\sigma(n-i+1)}) = p_\sigma(x_{\sigma^*(i)}), \quad \forall i = 1,\dots,n.\end{aligned}$$

Therefore $P^*_{\sigma^*} = P_\sigma \ \forall \sigma \in S_n$.

Sufficient condition: It suffices to prove that $g^*(\bar{A}) = 1 - g(A) \ \forall A \subseteq X$. For convenience, and without it being a restriction, let

$$A = \{x_1, x_2, \dots, x_i\} \quad \text{and} \quad \bar{A} = \{x_{i+1}, \dots, x_n\}.$$

Consider the dual permutations $\sigma = (1, 2, \dots, n)$ and $\sigma^* = (n, n-1, \dots, 1)$.

Then

$$
\begin{aligned}
g(A) &= g(\{x_1, \ldots, x_i\}) \\
&= g(\{x_1\}) + \sum_{j=2}^{i} (g(\{x_1, \ldots, x_j\}) - g(\{x_1, \ldots, x_{j-1}\})) \\
&= \sum_{j=1}^{i} p_\sigma(x_j) = P_\sigma(A), \\
g^*(\bar{A}) &= g^*(\{x_{i+1}, \ldots, x_n\}) \\
&= 1 - (g^*(\{x_1, \ldots, x_n\}) - g^*(\{x_{i+1}, \ldots, x_n\})) \\
&= 1 - \sum_{j=1}^{i} (g^*(\{x_j, \ldots, x_n\}) - g^*(\{x_{j+1}, \ldots, x_n\})) \\
&= 1 - \sum_{j=1}^{i} p^*_{\sigma^*}(x_j) = 1 - P^*_{\sigma^*}(A).
\end{aligned}
$$

Because of the hypothesis, $P_\sigma(A) = P^*_{\sigma^*}(A)$, and hence $g^*(\bar{A}) = 1 - g(A)$. □

If we accept that a fuzzy measure and its dual measure contain the same information, but codified in a different way, the above mentioned result could be interpreted by saying that the $n!$ associated probabilities contain the information and that the different orderings are different codifications of this information.

A fuzzy measure is said to be autodual if it coincides with its dual measure, that is, if $g^*(A) = g(A)$ $\forall A \subseteq X$. Starting from the last proposition, autoduality of a fuzzy measure is characterized in terms of its associated probabilities:

Corollary 3.1. *A fuzzy measure is autodual if and only if the probabilities associated to each permutation and to its dual permutation coincide.*

Another interesting case is when the fuzzy measure is a probability; it is the case when all associated probabilities are equal:

Proposition 3.3. *A fuzzy measure g is a probability measure if and only if its $n!$ associated probabilities coincide.*

Proof. The necessary condition is obvious because of the additivity of probability measures and of the definition of associated probabilities to a fuzzy measure.

To prove the sufficient condition, we denote P as the only associated probability. Given any subset $A \subseteq X$, we can deduce $g(A) = P_{\sigma_0}(A)$ for some permutation $\sigma_0 \in S_n$; as for any permutation $P(A) = P_\sigma(A)$ $\forall A \subseteq X$, it is obvious that $g(A) = P(A)$, and g is a probability. □

The case of capacities of order two is the most interesting one:

Proposition 3.4. *Let (g, g^*) be a pair of dual fuzzy measures. Then g is a lower capacity of order two (g^* is an upper capacity of order two respectively) if and* *only if*

$$g(A)=\min_{\sigma\in S_n} P_\sigma(A) \quad \forall A\subseteq X$$

$$(g^*(A)=\max_{\sigma\in S_n} P_\sigma(A) \quad \forall A\subseteq X, \quad \textit{respectively}).$$

Proof. We will only prove the case of lower capacities of order two. The proof for the upper capacities of order two is analogous.

Sufficient condition:

$$g(A)=\min_{\sigma\in S_n} P_\sigma(A)=\min_{\sigma\in S_n}\sum_{x_i\in A} p_\sigma(x_i)\leq \sum_{x_i\in A} p_\sigma(x_i),\ \forall\sigma\in S_n.$$

Let us suppose that

$$A\cap B=\{x_{i_1},\ldots,x_{i_r}\},\qquad \bar{A}\cap B=\{x_{j_1},\ldots,x_{j_s}\},$$
$$A\cap\bar{B}=\{x_{k_1},\ldots,x_{k_t}\}.$$

Consider a permutation $\tau\in S_n$ such that

$$\tau(1)=i_1,\ \ldots,\ \tau(r)=i_r,$$
$$\tau(r+1)=j_1,\ \ldots,\ \tau(r+s)=j_s,$$
$$\tau(r+s+1)=k_1,\ \ldots,\ \tau(r+s+t)=k_t,$$

that is, an ordering of elements of X such that the first places are taken up by elements of $A\cap B$, followed by the $\bar{A}\cap B$ ones, later the $A\cap\bar{B}$ ones, and finally, elements of $\overline{A\cup B}$. Then

$$\begin{aligned}
P_\tau(A)&=\sum_{x\in A} p_\tau(x)\\
&=g(\{x_{i_1}\})+g(\{x_{i_1},x_{i_2}\})-g(\{x_{i_1}\})\\
&\quad+\cdots+g(\{x_{i_1},\ldots,x_{i_r}\})-g(\{x_{i_1},\ldots,x_{i_{r-1}}\})\\
&\quad+g(\{x_{i_1},\ldots,x_{i_r},x_{j_1},\ldots,x_{j_s},x_{k_1}\})-g(\{x_{i_1},\ldots,x_{i_r},x_{j_1},\ldots,x_{j_s}\})\\
&\quad+\cdots+g(\{x_{i_1},\ldots,x_{i_r},x_{j_1},\ldots,x_{j_s},x_{k_1},\ldots,x_{k_t}\})\\
&\quad-g(\{x_{i_1},\ldots,x_{i_r},x_{j_1},\ldots,x_{j_s},x_{k_1},\ldots,x_{k_{t-1}}\})\\
&=g(\{x_{i_1},\ldots,x_{i_r}\})-g(\{x_{i_1},\ldots,x_{i_r},x_{j_1},\ldots,x_{j_s}\})\\
&\quad+g(\{x_{i_1},\ldots,x_{i_r},x_{j_1},\ldots,x_{j_s},x_{k_1},\ldots,x_{k_t}\})\\
&=g(A\cap B)-g(B)+g(A\cup B).
\end{aligned}$$

Therefore

$$g(A)\leq g(A\cap B)-g(B)+g(A\cup B),$$

and g is a lower capacity of order two.

Necessary condition: First, we will prove that if g is a lower capacity of order two, then

$$g(A)\leq\min_{\sigma\in S_n} P_\sigma(A),\quad \forall A\subseteq X.$$

Let us suppose that $A = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$. Then

$$P_\sigma(A) = \sum_{j=1}^{r} p_\sigma(x_{i_j}) = \sum_{j=1}^{r} (g(B_{\sigma i_j} \cup \{x_{i_j}\}) - g(B_{\sigma i_j})),$$

where $B_{\sigma i_j} = \{x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(h_{i_j}-1)}\}$, and $\sigma(h_{i_j}) = i_j$.

Without losing generality, we can suppose that

$$\sigma^{-1}(i_r) < \sigma^{-1}(i_{r-1}) < \cdots < \sigma^{-1}(i_1).$$

If we define the sets A_{i_j} by

$$A_{i_j} = \{x_{i_j}, x_{i_{j+1}}, \ldots, x_{i_r}\}, \quad j = 1, \ldots, r,$$

then

$$B_{\sigma i_j} \cup A_{i_j} = B_{\sigma i_j} \cup \{x_{i_j}\},$$
$$B_{\sigma i_j} \cap A_{i_j} = \{x_{i_{j+1}}, \ldots, x_{i_r}\} = A_{i_{j+1}}, \quad j = 1, \ldots, r-1,$$
$$B_{\sigma i_r} \cup A_{i_r} = B_{\sigma i_r} \cup \{x_{i_r}\}, \qquad B_{\sigma i_r} \cap A_{i_r} = \emptyset.$$

Thus

$$\sum_{j=1}^{r} g(B_{\sigma i_j} \cup \{x_{i_j}\}) \geq \sum_{j=1}^{r} g(B_{\sigma i_j}) + \sum_{j=1}^{r} g(A_{i_j}) - \sum_{j=1}^{r-1} g(A_{i_{j+1}})$$
$$= \sum_{j=1}^{r} g(B_{\sigma i_j}) + \sum_{j=1}^{r} g(A_{i_j}) - \sum_{j=2}^{r} g(A_{i_j}) = \sum_{j=1}^{r} g(B_{\sigma i_j}) + g(A_{i_1}).$$

Therefore

$$g(A_{i_1}) = g(A) \leq \sum_{j=1}^{r} (g(B_{\sigma i_j} \cup \{x_{i_j}\}) - g(B_{\sigma i_j})) = P_\sigma(A).$$

So,

$$g(A) \leq \min_{\sigma \in S_n} P_\sigma(A), \quad \forall A \subseteq X.$$

As a permutation $\sigma_0 \in S_n$ always exists such that $g(A) = P_{\sigma_0}(A)$, it is obvious that

$$g(A) = \min_{\sigma \in S_n} P_\sigma(A), \quad \forall A \subseteq X. \qquad \square$$

So, the main characteristic of a capacity of order two is that it only depends on the probabilities associated to such a measure, but does not depend on the permutations that generate them: we can regenerate the initial fuzzy measure by only knowing its associated probabilities, without necessity to know the corresponding permutations. This characteristic makes the use of capacities of order two by means of associated probabilities specially easy.

4. Properties of the monotone expectation in relation to the associated probabilities

The monotone expectation satisfies interesting properties in relation to the associated probabilities. One of them is the one we have used to justify our definition. We can express it in the following way:

Given a fuzzy measure g, for each non-negative function h, there exists a permutation $\sigma_h \in S_n$ and an associated probability P_{σ_h} such that

$$E_g(h) = E_{P_{\sigma_h}}(h) = \int_X h \, dP_{\sigma_h}.$$

Starting from this property the following result is evident and valid for every fuzzy measure:

Proposition 4.1. *If P_σ, $\sigma \in S_n$, are the associated probabilities to a fuzzy measure g, then for every $h: X \to \mathbb{R}_0^+$, it holds*

$$\min_{\sigma \in S_n} E_{P_\sigma}(h) \leq E_g(h) \leq \max_{\sigma \in S_n} E_{P_\sigma}(h).$$

Proof. As $\forall h: X \to \mathbb{R}_0^+ \; \exists \sigma_h \in S_n$ such that $E_g(h) = E_{P_\sigma}(h)$, it is evident that

$$\min_{\sigma \in S_n} E_{P_\sigma}(h) \leq E_g(h) \leq \max_{\sigma \in S_n} E_{P_\sigma}(h). \qquad \square$$

In general, that is all we can assure. Nevertheless, in the particular case of capacities of order two, one of the above bounds is necessarily reached by the monotone expectation value; moreover this fact is a characterization for this class of measures:

Proposition 4.2. *A necessary and sufficient condition for a pair of dual fuzzy measures (g, g^*) to be lower and upper capacities of order two respectively is that, $\forall h: X \to \mathbb{R}_0^+$,*

$$E_g(h) = \min_{\sigma \in S_n} E_{P_\sigma}(h) \quad \textit{and/or} \quad E_{g^*}(h) = \max_{\sigma \in S_n} E_{P_\sigma}(h).$$

Proof. *Necessary condition:* If g and g^* are capacities of order two, by Proposition 3.4,

$$g(A) = \min_{\sigma \in S_n} P_\sigma(A) \quad \text{and} \quad g^*(A) = \max_{\sigma \in S_n} P_\sigma(A) \quad \forall A \subseteq X,$$

and it is obvious that

$$g(A) \leq P_\sigma(A) \leq g^*(A) \;\; \forall A \subseteq X, \;\; \forall \sigma \in S_n.$$

So, by the monotonicity of monotone expectation,

$$E_g(h) \leq E_{P_\sigma}(h) \leq E_{g^*}(h) \quad \forall h: X \to \mathbb{R}_0^+.$$

Therefore

$$E_g(h) \leq \min_{\sigma \in S_n} E_{P_\sigma}(h), \qquad E_{g^*}(h) \geq \max_{\sigma \in S_n} E_{P_\sigma}(h).$$

Proposition 4.1 proves the opposite inequalities.

Sufficient condition: The hypothesis assures us that, for each $A \subseteq X$,

$$g(A) = E_g(I_A) = \min_{\sigma \in S_n} E_{P_\sigma}(I_A) = \min_{\sigma \in S_n} P_\sigma(A),$$

and we deduce that g is a lower capacity of order two from Proposition 3.4.

The result is similarly proved for g^*. □

The result we have proved permits us to establish another characterization, already known (see Huber [5]), of capacities of order two in a very simple way.

Proposition 4.3. (a) *g is a lower capacity of order two if and only if*

$$E_g(h_1 + h_2) \geq E_g(h_1) + E_g(h_2), \quad \forall h_1, h_2 : X \to \mathbb{R}_0^+.$$

(b) *g^* is an upper capacity of order two if and only if*

$$E_{g^*}(h_1 + h_2) \leq E_{g^*}(h_1) + E_{g^*}(h_2), \quad \forall h_1, h_2 : X \to \mathbb{R}_0^+.$$

Proof. (a) *Necessary condition:* If g is a lower capacity of order two, then from Proposition 4.2,

$$E_g(h_1 + h_2) = \min_{\sigma \in S_n} E_{P_\sigma}(h_1 + h_2) = \min_{\sigma \in S_n} (E_{P_\sigma}(h_1) + E_{P_\sigma}(h_2))$$
$$\geq \min_{\sigma \in S_n} E_{P_\sigma}(h_1) + \min_{\sigma \in S_n} E_{P_\sigma}(h_2) = E_g(h_1) + E_g(h_2), \quad \forall h_1, h_2 : X \to \mathbb{R}_0^+.$$

Sufficient condition: As $E_g(h_1 + h_2) \geq E_g(h_1) + E_g(h_2)\ \forall h_1, h_2 : X \to \mathbb{R}_0^+$, if we make $h_1 = I_A$ and $h_2 = I_B$, it holds

$$g(A) + g(B) = E_g(I_A) + E_g(I_B) \leq E_g(I_A + I_B) = E_g(I_{A \cup B} + I_{A \cap B}).$$

It is easy to verify that

$$E_g(I_{A \cup B} + I_{A \cap B}) = E_g(I_{A \cup B}) + E_g(I_{A \cap B}) = g(A \cup B) + g(A \cap B).$$

This fact should not surprise us because functions $I_{A \cap B}$ and $I_{A \cup B}$ rank their values in the same way, and for this type of functions, the monotone expectation is an additive functional.

Therefore g is a lower capacity of order two.

(b) This is similarly proved. □

5. Conclusions

We have been able to represent all fuzzy measures as ordered sets of probabilities, basing ourselves on the behaviour of the monotone expectation functional. We have also studied some fuzzy measures and monotone expectation properties in the light of the associated probabilities.

We think this representation of fuzzy measures by associated probabilities provides a new perspective on them. For further works, we propose to study concepts such as distance, inclusion relations and combination methods for fuzzy measures from the concept of associated probabilities.

References

[1] M.J. Bolaños, M.T. Lamata and S. Moral, La esperanza monotona: una generalización de la esperanza probabilista, in: *Actas de la XV Reunión Nacional de Estadística e Investigación Operativa*, Asturias (1985) 57–64.

[2] L.M. de Campos, Caracterización y estudio de medidas e integrales difusas a partir de probabilidades, Thesis, Universidad de Granada (1987).

[3] G. Choquet, Theory of capacities, *Ann. Inst. Fourier* **5** (1953) 131–295.

[4] A.P. Dempster, Upper and lower probabilities induced by a multivaluated mapping, *Ann. Math. Statist.* **38** (1967) 325–339.

[5] P.J. Huber, *Robust Statistics* (Wiley, New York, 1981).

[6] R. Kruse, A note on λ-additive fuzzy measures, *Fuzzy Sets and Systems* **10** (1982) 309–313.

[7] M.T. Lamata, Modelos de decisión con información general, Thesis, Universidad de Granada (1985).

[8] G. Shafer, *A Mathematical Theory of Evidence* (Princeton Univ. Press, Princeton, NJ, 1976).

[9] M. Sugeno, Theory of fuzzy integrals and its applications, Thesis, Tokio Inst. of Technology (1974).

[10] S. Weber, Two integrals and some modified versions. Critical remarks, *Fuzzy Sets and Systems* **20** (1986) 97–105.

[11] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems* **1** (1978) 3–28.