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CONVERGENCE PROPERTIES OF THE MONOTONE EXPECTATION AND ITS APPLICATION TO THE EXTENSION OF FUZZY MEASURES

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Abstract: We define a functional on fuzzy measures (called monotone expectation) based on Choquet's integral operator, and we study its convergence properties. The monotone expectation is used to extend fuzzy measures to fuzzy subsets. Also Sugeno's bound for the difference between his integral and the mathematical expectation is generalized.

Keywords: Fuzzy measure; fuzzy integral; monotone expectation.

1. Introduction

Since the publication of Zadeh's [12] pioneering work, several new contributions have appeared on the problem of evaluating fuzzy sets, particularly regarding the extension of defined measures from crisp subsets of the referential to fuzzy subsets.

While Zadeh presented a natural generalization of the concept of probability to fuzzy events by integration of the membership function, it was Sugeno [9] who defined the first functional (a fuzzy integral) for the extension of fuzzy measures. Although probability was considered to reflect a particular case of fuzzy measure, Sugeno's definition differs from Zadeh's being based, as numerous authors have shown in previous analyses and characterizations (Batle and Trillas [1], Ralescu and Adams [7], Kandel [4]), on a max–min or 'fixed point' approach.

Various functionals have been proposed to deal with particular types of fuzzy measures (Kruse [5], Nguyen [6], Smets [8], Weber [11]). Some of these authors have suggested the usefulness of Choquet's integral operator [3], defined originally for capacities, in the extension of measures to fuzzy sets. In the present study we describe the application of this approach to any fuzzy measure by defining the monotone expectation as a functional applicable to non-negative functions, particularly to membership functions of fuzzy subsets. We furthermore characterize the key role of monotone expectation as the natural generalization of the Lebesgue integral to the fuzzy context. After analyzing the basic properties of monotone expectation in Section 2, Section 3 presents proofs of the theorems of monotone and bounded convergence, equivalent to Levi's and Lebesgue's theorems respectively, in the context of classic integration. Finally, Section 4 deals with the problem, as mentioned above, of extending fuzzy measures to fuzzy subsets. The extension given by monotone expectation is shown to provide fuzzy measures; furthermore, Sugeno's bound (proved by the author for the particular case of probability) is found to be valid in the much more general field of fuzzy measures, in which it is shown to represent the difference between monotone expectation and the fuzzy integral. Our results seem to indicate that the formal similarities analyzed by Sugeno between his integral and Lebesgue's should be more appropriately established between Sugeno's integral and monotone expectation.

2. Definition and basic properties

In order to define the functional monotone expectation we need a fuzzy measure defined on an appropriate class of crisp subsets of a universe. We adopt Sugeno's definition of fuzzy measure [9], that is a bounded, monotone and continuous for monotone sequences set function.

We consider the space $(X, \mathscr{B})$, where X is an arbitrary universe and $\mathscr{B}$ is a σ-algebra on X (although for definition it suffices to consider a monotone class, sufficiently rich properties are only obtainable with σ-algebras).

Definition 2.1. A fuzzy measure g on $(X, \mathscr{B})$ is a valuation $g: \mathscr{B} \to [0, 1]$ satisfying

(i) $g(\emptyset) = 0$, $g(X) = 1$.

(ii) If $A, B \in \mathscr{B}$, and $A \subseteq B$, then $g(A) \leqslant g(B)$.

(iii) If $\{A_n\}$ is a monotone sequence of elements in $\mathscr{B}$, then

$$\lim_{n\to\infty} g(A_n) = g(\lim_{n\to\infty} A_n).$$

Fuzzy measures are known to be generalizations of probability measures, where additivity is replaced by the weaker condition of monotonicity.

For this reason, it seems suitable to have a functional that extends the classical idea of mathematical expectation on probability measures to fuzzy measures. It is possible to establish such a generalization in terms of the integral defined by Choquet [3] for capacities. We will call this integral monotone expectation because this name better reflects its basic properties.

Definition 2.2. Given a non-negative $\mathscr{B}$-measurable function $h: X \to \mathbb{R}_0^+$ and a fuzzy measure g on $(X, \mathscr{B})$, the monotone expectation (m.e.) of h with respect to g is the Lebesgue integral

$$E_g(h) = \int_0^{+\infty} g(\{x \in X \mid h(x) \geqslant \alpha\})\, d\alpha = \int_0^{+\infty} g(H_\alpha)\, d\alpha$$

where H_α are the h level sets, and $g(H_\alpha)$ is called measure function of h.

Before studying the properties of m.e., it is necessary to verify the existence of this functional.

If the universe X is finite, m.e. always exists for each non-negative function h, as the measure function of h is a simple function. In the general case, the existence of m.e. is not assured.

Obviously, h must be $\mathscr{B}$-measurable ($H_\alpha \in \mathscr{B}\ \forall \alpha \geq 0$) and $g(H_\alpha)$ must be Lebesgue measurable. The last condition presents no problem, since $g(H_\alpha)$ is a bounded and non-increasing function. However it is not sufficient to assure integrability (i.e., the existence of a finite integral).

When h is an upper bounded function, it is possible to assure the existence of $E_g(h)$:

As $k > 0$ exists such that $h(x) \leq k\ \forall x \in X$, then $H_\alpha = \emptyset\ \forall \alpha > k$ and $g(H_\alpha) = 0\ \forall \alpha > k$. Therefore

$$E_g(h) = \int_0^{+\infty} g(H_\alpha)\,\mathrm{d}\alpha = \int_0^k g(H_\alpha)\,\mathrm{d}\alpha,$$

and this last integral always exists because $g(H_\alpha)$ is a monotone non-increasing and bounded function.

Hence the existence of m.e. is guaranteed for $\mathscr{B}$-measurable membership functions of fuzzy subsets in X.

As is usual in measure theory, we will say h is a g-integrable function when $E_g(h) < \infty$.

What follows are the most important properties of m.e.:

Proposition 2.1 (h-monotonicity). *Let $h_1, h_2 : X \to \mathbb{R}_0^+$ be two g-integrable functions with respect to the fuzzy measure space $(X, \mathscr{B}, g)$. If*

$$h_1(x) \leq h_2(x) \quad \forall x \in X,$$

then

$$E_g(h_1) \leq E_g(h_2).$$

Proof. Since $h_1(x) \leq h_2(x)\ \forall x \in X$, it is clear that $H_{1\alpha} \subseteq H_{2\alpha}\ \forall \alpha \geq 0$, and then

$$g(H_{1\alpha}) \leq g(H_{2\alpha}) \quad \forall \alpha \geq 0.$$

Monotonicity of the Lebesgue integral suffices to prove the result. □

Proposition 2.2 (g-monotonicity). *Let g_1, g_2 be fuzzy measures defined on $(X, \mathscr{B})$, and let $h : X \to \mathbb{R}_0^+$ a g_i-integrable function, $i = 1, 2$. If*

$$g_1(A) \leq g_2(A) \quad \forall A \in \mathscr{B}$$

then

$$E_{g_1}(h) \leq E_{g_2}(h).$$

Proof. As $g_1(H_\alpha) \leq g_2(H_\alpha)\ \forall \alpha \geq 0$, again monotonicity of the Lebesgue integral is sufficient to prove our claim. □

Proposition 2.3. *For every constant function* ($h(x)=c\geqslant 0\ \forall x\in X$),

$$E_g(h)=c$$

for each fuzzy measure g *on* $(X, \mathscr{B})$.

Proof. If $\alpha\leqslant c$ then $g(H_\alpha)=g(X)=1$. If $\alpha>c$ then $g(H_\alpha)=g(\emptyset)=0$. Therefore

$$\int_0^{+\infty} g(H_\alpha)\,\mathrm{d}\alpha=\int_0^{c}\mathrm{d}\alpha=c. \qquad \square$$

Proposition 2.4. *For every* g*-integrable function* $h:X\to\mathbb{R}_0^+$ *with respect to* $(X, \mathscr{B}, g)$, $a+bh$ *is a* g*-integrable function for each* $a, b\in\mathbb{R}_0^+$, *and*

$$E_g(a+bh)=a+bE_g(h).$$

Proof.

$$\begin{aligned}E_g(a+bh)&=\int_0^{+\infty} g(\{x\mid a+bh(x)\geqslant\alpha\})\,\mathrm{d}\alpha\\&=\int_0^{a} g(\{x\mid a+bh(x)\geqslant\alpha\})\,\mathrm{d}\alpha+\int_a^{\infty} g\left(\left\{x\mid h(x)\geqslant\frac{\alpha-a}{b}\right\}\right)\mathrm{d}\alpha.\end{aligned}$$

The first addend is equal to

$$\int_0^a \mathrm{d}\alpha=a.$$

If we change the variable in the second addend to $\gamma=(\alpha-a)/b$ then

$$\int_a^{+\infty} g\left(\left\{x\mid h(x)\geqslant\frac{\alpha-a}{b}\right\}\right)\mathrm{d}\alpha=b\int_0^{+\infty} g(\{x\mid h(x)\geqslant\gamma\})\,\mathrm{d}\gamma=bE_g(h).$$

Therefore $E_g(a+bh)=a+bE_g(h)$. $\square$

Proposition 2.5. *If* h *is the characteristic function of a crisp set* $A\in\mathscr{B}$, *then*

$$E_g(h)=g(A),$$

for every fuzzy measure g *on* $(X, \mathscr{B})$.

Proof.

$$E_g(h)=\int_0^{+\infty} g(H_\alpha)\,\mathrm{d}\alpha=\int_0^{1} g(A)\,\mathrm{d}\alpha=g(A). \qquad \square$$

Proposition 2.6. *The monotone expectation is an extension to fuzzy measures of the mathematical expectation for probabilities.*

Proof. Let P be a probability measure on $(X, \mathscr{B})$ and let $h:X\to\mathbb{R}_0^+$ be a

P-integrable function. Then

$$\int_X h\,\mathrm{d}P = \int_0^{+\infty} P(\{x \mid h(x) \geq \alpha\})\,\mathrm{d}\alpha = \int_0^{+\infty} P(H_\alpha)\,\mathrm{d}\alpha = E_P(h). \qquad \square$$

The above propositions show how m.e. maintains some of the most important properties of mathematical expectation. However, m.e. in general is not a linear functional, as it is only additive when the fuzzy measure considered is a probability:

Proposition 2.7. *$E_g(h_1 + h_2) = E_g(h_1) + E_g(h_2)$ for all g-integrable functions $h_1, h_2 : X \to \mathbb{R}_0^+$ if and only if the fuzzy measure g is a probability measure.*

Proof. The sufficient condition is obvious from Proposition 2.6. To prove the necessary condition, we consider the characteristic functions I_A and I_B of crisp sets $A, B \in \mathcal{B}$, such that $A \cap B = \emptyset$.

Using the present hypothesis and Proposition 2.5, there holds

$$g(A \cup B) = E_g(I_{A\cup B}) = E_g(I_A + I_B) = E_g(I_A) + E_g(I_B) = g(A) + g(B).$$

which is a necessary and sufficient condition for g to be a probability. $\square$

Proofs of above propositions in the finite case can be found in Bolaños et al. [2].

As we have seen, the properties of m.e. make it more similar to mathematical expectation than other functionals, such as the well-known Sugeno integral. This latter does not possess acceptable properties of linearity, so, when these properties are of interest, it seems better to use m.e. than Sugeno's integral.

Sugeno's integral and monotone expectation valuate functions starting from fuzzy measures in different way. The first corresponds to a max–min approach, whereas the second extends classical ideas of integration to a non-additive context.

3. Convergence properties of the monotone expectation

Convergence properties are basic to the study of fuzzy measures and integrals. In this section we describe some important results of this kind for m.e. First, the validity of Levi's monotone convergence theorems is proved. These theorems allow us to use m.e. to extend fuzzy measures to fuzzy subsets of X. On the other hand, we obtain results similar to Lebesgue's dominated convergence theorem, which fixes conditions for convergence in the absence of monotonicity.

Previously we shall prove a lemma that allows us to use both strong and weak α-cuts in the definition of m.e.:

Lemma 3.1. *Let $h : X \to \mathbb{R}_0^+$ be a g-integrable function with respect to the fuzzy* *measure space (X, B, g). There holds*

$$E_g(h)=\int_0^{+\infty} g(H_\alpha)\,\mathrm{d}\alpha=\int_0^{+\infty} g(H_\alpha^-)\,\mathrm{d}\alpha,$$

where $H_\alpha=\{x\in X \mid h(x)\geq\alpha\}$, and $H_\alpha^-=\{x\in X \mid h(x)>\alpha\}$.

Proof. If we denote $f(\alpha)=g(H_\alpha)$ and $f'(\alpha)=g(H_\alpha^-)$, then

$$\int_0^{+\infty} f'(\alpha)\,\mathrm{d}\alpha\leq\int_0^{+\infty} f(\alpha)\,\mathrm{d}\alpha.$$

Moreover, $f(\alpha+\varepsilon)\leq f'(\alpha)\quad\forall\varepsilon>0, \forall\alpha>0$, and hence

$$\int_0^{+\infty} f(\alpha+\varepsilon)\,\mathrm{d}\alpha\leq\int_0^{+\infty} f'(\alpha)\,\mathrm{d}\alpha\quad\forall\varepsilon>0.$$

If we substitute the variable γ for $\alpha+\varepsilon$ in the first integral, then

$$\int_\varepsilon^{+\infty} f(\gamma)\,\mathrm{d}\gamma\leq\int_0^{+\infty} f'(\alpha)\,\mathrm{d}\alpha\quad\forall\varepsilon>0.$$

Therefore

$$\int_0^{+\infty} f(\gamma)\,\mathrm{d}\gamma\leq\int_0^{+\infty} f'(\alpha)\,\mathrm{d}\alpha,$$

and we obtain the equality between the two integrals. □

Proposition 3.1. *Let $h_n:X\to\mathbb{R}_0^+$ be a monotone non-decreasing sequence of functions which converges to $h:X\to\mathbb{R}_0^+$. If h_n are g-integrable functions with respect to $(X, \mathscr{B}, g)$ and, either h is g-integrable too or $\lim_{n\to\infty}E_g(h_n)<\infty$ exists, then h is g-integrable and*

$$\lim_{n\to\infty} E_g(h_n)=E_g(h).$$

Proof. We denote $(H_\alpha^-)_n=\{x\in X \mid h_n(x)>\alpha\}$ and $H_\alpha^-=\{x\in X \mid h(x)>\alpha\}$. It is clear that $\{(H_\alpha^-)_n\}$ is a non-decreasing sequence which converges to $\bigcup_{n=1}^\infty (H_\alpha^-)_n$. We prove that this limit is equal to H_α^-:

Let $x\in\bigcup_{n=1}^\infty (H_\alpha^-)_n$; then n_0 exists such that $x\in(H_\alpha^-)_{n_0}$, hence $h_{n_0}(x)>\alpha$. As $\{h_n\}$ is a non-decreasing sequence,

$$\lim_{n\to\infty} h_n(x)=h(x)\geq h_{n_0}(x),$$

and then $x\in H_\alpha^-$. If there exists $x\in H_\alpha^-$ such that $x\notin\bigcup_{n=1}^\infty (H_\alpha^-)_n$, then $x\notin(H_\alpha^-)_n\ \forall n\in\mathbb{N}$, implying $h_n(x)\leq\alpha\ \forall n\in\mathbb{N}$ and hence $\lim_{n\to\infty}h_n(x)=h(x)\leq\alpha$, so that $x\notin H_\alpha^-$, in contradiction to our hypothesis. Therefore

$$\lim_{n\to\infty} (H_\alpha^-)_n=\bigcup_{n=1}^\infty (H_\alpha^-)_n=H_\alpha^-,\quad\forall\alpha>0.$$

From the continuity of fuzzy measures, it follows that

$$\lim_{n\to\infty} g((H_\alpha^-)_n) = g(H_\alpha^-) \quad \forall\alpha > 0,$$

and thus convergence of the measure functions sequence is proved.

Applying the monotone convergence theorem for Lebesgue's integral and the above lemma, there holds

$$\lim_{n\to\infty} E_g(h_n) = \lim_{n\to\infty} \int_0^{+\infty} g((H_\alpha^-)_n)\,\mathrm{d}\alpha = \int_0^{+\infty} g(H_\alpha^-)\,\mathrm{d}\alpha = E_g(h). \qquad \square$$

Analogous results are obtained for non-increasing sequences:

Proposition 3.2. *Let $h_n : X \to \mathbb{R}_0^+$ be a monotone non-increasing sequence of functions which converges to $h : X \to \mathbb{R}_0^+$. If h_n are g-integrable functions with respect to $(X, \mathscr{B}, g)$, then h is g-integrable too, and*

$$\lim_{n\to\infty} E_g(h_n) = E_g(h).$$

Proof. We denote $(H_\alpha)_n = \{x \in X \mid h_n(x) \geq \alpha\}$ and $H_\alpha = \{x \in X \mid h(x) \geq \alpha\}$. Obviously $\{(H_\alpha)_n\}$ is a non-decreasing sequence which converges to $\bigcap_{n=1}^{\infty} (H_\alpha)_n$. We prove this limit coincides with H_α:

$\forall x \in \bigcap_{n=1}^{\infty} (H_\alpha)_n$, $x \in (H_\alpha)_n$ $\forall n \in \mathbb{N}$, and $h_n(x) \geq \alpha$ $\forall n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} h_n(x) = h(x) \geq \alpha$ and $x \in H_\alpha$.

If $x \notin \bigcap_{n=1}^{\infty} (H_\alpha)_n$, then $\exists n_0 \in \mathbb{N}$ such that $x \notin (H_\alpha)_{n_0}$. Then $\exists n_0$ such that $h_{n_0}(x) < \alpha$ which implies $h(x) \leq h_{n_0}(x) < \alpha$ and thus $x \notin H_\alpha$.

Therefore

$$\bigcap_{n=1}^{\infty} (H_\alpha)_n = H_\alpha \quad \forall\alpha > 0.$$

Again, by virtue of the convergence properties of Lebesgue's integral and the continuity of fuzzy measures, it can be concluded that

$$\lim_{n\to\infty} E_g(h_n) = \lim_{n\to\infty} \int_0^{+\infty} g((H_\alpha)_n)\,\mathrm{d}\alpha = \int_0^{+\infty} g(H_\alpha)\,\mathrm{d}\alpha = E_g(h). \qquad \square$$

Several previous lemmas are required to extend the convergence theorem of Lebesgue:

Lemma 3.2. *Let $h, k : X \to \mathbb{R}_0^+$ be two functions satisfying:*

(a) *$h(x) \leq k(x)$ $\forall x \in X$.*

(b) *h is a $\mathscr{B}$-measurable function.*

(c) *k is g-integrable with respect to $(X, \mathscr{B}, g)$.*

Then h is also g-integrable with respect to $(X, \mathscr{B}, g)$.

Proof. If we denote the α-cuts of h and k as H_α and K_α respectively, then $g(H_\alpha)$ is a Lebesgue measurable function because h is $\mathscr{B}$-measurable, and $g(K_\alpha)$ is Lebesgue integrable since k is a g-integrable function.

As $0 \leqslant g(H_\alpha) \leqslant g(K_\alpha)\ \forall \alpha > 0$, $g(H_\alpha)$ is also Lebesgue integrable and hence h is g-integrable. □

Lemma 3.3. *Let $h_n : X \to \mathbb{R}_0^+$ be a sequence of g-integrable functions with respect to $(X, \mathscr{B}, g)$ such that $h_n(x) \leqslant k(x)\ \forall x \in X$, $\forall n \in \mathbb{N}$, where k is g-integrable too. Then*

$$E_g\left(\limsup_{n\to\infty} h_n\right) \geqslant \limsup_{n\to\infty} E_g(h_n).$$

Proof. We define functions p_n by

$$p_n(x) = \sup_{i \geqslant n} h_i(x) \quad \forall x \in X.$$

$\{p_n\}$ is a non-increasing sequence of functions which converges to $\limsup_{n\to\infty} h_n$. Since $p_n(x) \leqslant k(x)\ \forall x \in X$, $\forall n \in \mathbb{N}$, the functions p_n are g-integrable by Lemma 3.2. Applying Proposition 3.2 to the sequence $\{p_n\}$, we obtain

$$\lim_{n\to\infty} E_g(p_n) = E_g\left(\limsup_{n\to\infty} h_n\right).$$

As $h_n(x) \leqslant p_n(x)\ \forall x \in X$, $\forall n \in \mathbb{N}$, then $E_g(h_n) \leqslant E_g(p_n)\ \forall n \in \mathbb{N}$, and therefore

$$\limsup_{n\to\infty} E_g(h_n) \leqslant \limsup_{n\to\infty} E_g(p_n) = \lim_{n\to\infty} E_g(p_n) = E_g\left(\limsup_{n\to\infty} h_n\right). \qquad \square$$

Lemma 3.4. *Let $h_n : X \to \mathbb{R}_0^+$ be a sequence of g-integrable functions with respect to $(X, \mathscr{B}, g)$. There holds*

$$E_g\left(\liminf_{n\to\infty} h_n\right) \leqslant \liminf_{n\to\infty} E_g(h_n).$$

Proof. We define functions q_n by

$$q_n(x) = \inf_{i \geqslant n} h_i(x) \quad \forall x \in X.$$

$\{q_n\}$ is a non-decreasing sequence of functions which converges to $\liminf_{n\to\infty} h_n$. As $q_n(x) \leqslant h_n(x)\ \forall x \in X$, $\forall n \in \mathbb{N}$, again by Lemma 3.2, q_n are g-integrable functions, and $E_g(q_n) \leqslant E_g(h_n)\ \forall n \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty} E_g(q_n) \leqslant \liminf_{n\to\infty} E_g(h_n),$$

and, by Proposition 3.1,

$$E_g\left(\liminf_{n\to\infty} h_n\right) = \lim_{n\to\infty} E_g(q_n) \leq \liminf_{n\to\infty} E_g(h_n). \qquad \square$$

Now we can prove the following result:

Proposition 3.3. *Let $h_n : X \to \mathbb{R}_0^+$ be a sequence of $\mathcal{B}$-measurable functions which converges to $h : X \to \mathbb{R}_0^+$. If there exists a function k, g-integrable with respect to $(X, \mathcal{B}, g)$, such that*

$$h_n(x) \leq k(x) \quad \forall x \in X, \forall n \in \mathbb{N},$$

then h, h_n are g-integrable functions $\forall n \in \mathbb{N}$, and

$$\lim_{n\to\infty} E_g(h_n) = E_g(h).$$

Proof. Using Lemmas 3.3 and 3.4, we have

$$E_g\left(\liminf_{n\to\infty} h_n\right) \leq \liminf_{n\to\infty} E_g(h_n) \leq \limsup_{n\to\infty} E_g(h_n) \leq E_g\left(\limsup_{n\to\infty} h_n\right).$$

As $\lim_{n\to\infty} h_n = h$, it follows that $\liminf_{n\to\infty} h_n = \limsup_{n\to\infty} h_n = h$. Thus

$$E_g(h) = E_g\left(\lim_{n\to\infty} h_n\right) = \lim_{n\to\infty} E_g(h_n). \qquad \square$$

Corollary. *Let $(x, \mathcal{B}, g)$ be any fuzzy measure space, and let $\{A_n\}$ be a sequence of elements of $\mathcal{B}$ which converges to A. Then*

$$\lim_{n\to\infty} g(A_n) = g(A).$$

Proof. It suffices to apply the above proposition to the characteristic functions I_{A_n} and I_A. $\square$

This result assures us that continuity of any fuzzy measure for every sequence of measurable subsets can indeed be inferred from its monotonicity and continuity for monotone sequences.

4. Extension of fuzzy measures to fuzzy subsets

For fuzzy subsets of a universe X we can establish a definition of fuzzy measure just as was done in Definition 2.1 for crisp subsets of X. The concepts of fuzzy monotone class, fuzzy σ-algebra and similar properties to (i), (ii) and (iii) of this definition can be defined by direct extension of the corresponding crisp concepts to membership functions.

Given a fuzzy measure g on the crisp measurable space $(X, \mathscr{B})$, we attempt to use the monotone expectation to extend g to a fuzzy measure $g_{\sim}$ on the fuzzy measurable space $(X, \mathscr{B}_{\sim})$.

Proposition 4.1. *Let $(X, \mathscr{B}, g)$ be a fuzzy measure space. The valuation $g_{\sim}: \mathscr{B}_{\sim} \to [0, 1]$ defined by*

$$g_{\sim}(A_{\sim}) = E_g(\mu_A) \quad \forall A_{\sim} \in B_{\sim}$$

is a fuzzy measure which is an extension of g, where $B_{\sim}$ is the fuzzy σ-algebra of fuzzy subsets $A_{\sim}$ of X with g-integrable membership function μ_A.

Proof. Class $\mathscr{B}_{\sim}$ is a fuzzy σ-algebra because:

(a) If μ_A is g-integrable, $1 - \mu_A$ is also g-integrable.

(b) If $\forall n \in \mathbb{N}$, μ_{A_n} are g-integrable functions, $\sup_n \mu_{A_n}$ is g-integrable since it is a $\mathscr{B}$-measurable bounded function.

We will prove $g_{\sim}$ satisfies the conditions of a fuzzy measure:

(i) $g_{\sim}(\emptyset) = E_g(0_X) = 0$; $g_{\sim}(X) = E_g(1_X) = 1$.

(ii) If $A_{\sim}, B_{\sim} \in \mathscr{B}_{\sim}$ and $A_{\sim} \subseteq B_{\sim}$, then $\mu_A(x) \leq \mu_B(x)\ \forall x \in X$, and therefore

$$g_{\sim}(A_{\sim}) = E_g(\mu_A) \leq E_g(\mu_B) = g_{\sim}(B_{\sim}).$$

(iii) If $\{A_{n\sim}\}$ is a monotone sequence of elements of $\mathscr{B}_{\sim}$ which converges to $A_{\sim}$, then

$$\lim_{n\to\infty} \mu_{A_n}(x) = \mu_A(x) \quad \forall x \in X.$$

Applying Proposition 3.1 or 3.2, we have

$$\lim_{n\to\infty} g_{\sim}(A_{n\sim}) = \lim_{n\to\infty} E_g(\mu_{A_n}) = E_g(\mu_A) = g_{\sim}(A_{\sim}).$$

Therefore g is a fuzzy measure. Finally, if $A \in \mathscr{B}$, then

$$g_{\sim}(A) = E_g(I_A) = g(A),$$

and $g_{\sim}$ is a extension of g. □

In the same way as the crisp case, the fuzzy measure $g_{\sim}$ moreover is continuous for any sequence of fuzzy sets of $\mathscr{B}_{\sim}$:

Proposition 4.2. *Let $(X, \mathscr{B}, g)$ be a fuzzy measure space and $(X, \mathscr{B}_{\sim}, g_{\sim})$ the extended fuzzy measure space. If $\{A_{n\sim}\}$ is a sequence of elements of $\mathscr{B}_{\sim}$, which converges to $A_{\sim}$, then*

$$\lim_{n\to\infty} g_{\sim}(A_{n\sim}) = g_{\sim}(A_{\sim}).$$

Proof. It suffices to apply Proposition 3.3 for membership functions. □

In order to extend fuzzy measures to fuzzy subsets, the normally used choice until now is based on the use of Sugeno's integral, assigning each fuzzy subset the value:

$$\forall A_\sim \in \mathcal{B}_\sim \quad g_s(A_\sim) = \int \mu_A \circ g = \sup_{\alpha\in[0,1]} \{\alpha \wedge g(A_\alpha)\}.$$

The monotone expectation offers another alternative to perform this extension. As we have seen, it can be defined by

$$\forall A_\sim \in \mathcal{B}_\sim \quad g_\sim(A_\sim) = E_g(\mu_A) = \int_0^1 g(A_\alpha)\,\mathrm{d}\alpha.$$

Sugeno [9] proved that if we use a probability P as the measure, the difference in absolute value between P_s (Sugeno's extension) and $P_\sim$ (classical extension (Zadeh [12])) is smaller than or equal to $\frac{1}{4}$. This bound can be generalized to any fuzzy measure, where the role of mathematical expectation is played by monotone expectation.

Proposition 4.3. *Let $(X, \mathcal{B}, g)$ be a fuzzy measure space, and g_s and $g_\sim$ extensions of g by Sugeno's integral and the monotone expectation, respectively. There holds*

$$\forall A_\sim \in \mathcal{B}_\sim \quad |g_\sim(A_\sim) - g_s(A_\sim)| \leq g_s(A_\sim)(1 - g_s(A_\sim)).$$

Proof. Let $A_\sim \in \mathcal{B}_\sim$. By letting $G(\alpha) = g(A_\alpha)$, $c = g_s(A_\sim)$ and $b = g_\sim(A_\sim)$, and on the basis of results previously established by Sugeno ([9], Theorem 3.9), there holds

$$G(c^+) \leq c \leq G(c).$$

As G is a non-decreasing function, c can be interpreted as the value of α at the intersection of $G(\alpha)$ and bisectrix of the first quadrant, if G is continuous in c; otherwise, c is the only value verifying

$$G(\alpha) \geq c \ \ \forall \alpha < c \quad \text{and} \quad G(\alpha) \leq c \ \ \forall \alpha > c.$$

Consequently c is the area of the rectangle $[0, c] \times [0, 1]$, whereas b is the area below $G(\alpha)$, as illustrated in Figure 1.

Under these conditions, we have

$$\begin{aligned} b - c &= \int_0^1 G(\alpha)\,\mathrm{d}\alpha - c = \int_c^1 G(\alpha)\,\mathrm{d}\alpha - \left(c - \int_0^c G(\alpha)\,\mathrm{d}\alpha\right) \\ &\leq \int_c^1 G(\alpha)\,\mathrm{d}\alpha \leq \int_c^1 c\,\mathrm{d}\alpha = c(1-c). \end{aligned}$$

Furthermore,

$$\begin{aligned} b - c &= \int_c^1 G(\alpha)\,\mathrm{d}\alpha - \left(c - \int_0^c G(\alpha)\,\mathrm{d}\alpha\right) \geq -c + \int_0^c G(\alpha)\,\mathrm{d}\alpha \\ &\geq -c + \int_0^c c\,\mathrm{d}\alpha = -c + c^2 = -c(1-c). \end{aligned}$$

Therefore $|b - c| \leq c(1-c)$. $\square$

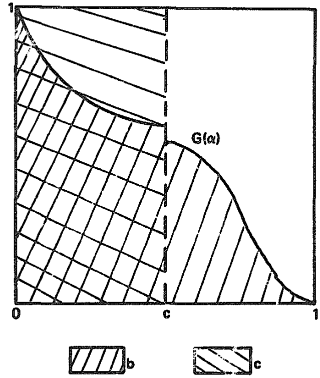

Fig. 1.

The generalization of Sugeno's bound can be obtained from the above proposition:

Corollary. *There holds*

$$\forall A_\sim \in \mathcal{B}_\sim \quad |g_\sim(A_\sim) - g_s(A_\sim)| \leqslant \tfrac{1}{4}.$$

Proof. This logically follows, as the function $x(1-x)$ reaches its maximum value at $\frac{1}{4}$ for $x_0 = \frac{1}{2}$. □

References

[1] N. Batle and E. Trillas, Entropy and fuzzy integral, *J. Math. Anal. Appl.* **69** (1979) 469–474.
[2] M.J. Bolaños, M.T. Lamata and S. Moral, La esperanza monótona: una generalización de la esperanza probabilística, in: *Actas XV Reunión Nacional de Estadística e Investigación Operativa*, Asturia (1985) 57–64.
[3] G. Choquet, Theory of capacitities, *Ann. Inst. Fourier* **5** (1953) 131–295.
[4] A. Kandel, On fuzzy statistics, in: M.M. Gupta, R.K. Ragade and R.R. Yager, Eds., *Advances in Fuzzy Sets Theory and Applications* (North-Holland, Amsterdam 1979) 181–198.
[5] R. Kruse, A note on λ-additive fuzzy measures, *Fuzzy Sets and Systems* **10** (1982) 309–313.
[6] H.T. Nguyen, Possibility measures and related topics, in: M.M. Gupta and E. Sanchez, Eds., *Fuzzy Information and Decision Processes* (North-Holland, Amsterdam, 1982) 197–201.
[7] D. Ralescu and G. Adams, The fuzzy integral, *J. Math. Anal. Appl.* **75** (1980) 562–570.
[8] P. Smets, The degree of belief in a fuzzy event, *Inform. Sci.* **25** (1981) 1–19.
[9] M. Sugeno, Theory of fuzzy integrals and its applications, Ph. D. Thesis, Tokio Inst. of Technology (1974).
[10] M. Sugeno and T. Mirofushi, Choquet's integral as an integral form for a general class of fuzzy measures, Presented at *IFSA'87*, Tokio (1987).
[11] S. Weber, Two integrals and some modified versions. Critical remarks, *Fuzzy Sets and Systems* **20** (1986) 97–105.
[12] L.A. Zadeh, Probability measures of fuzzy events, *J. Math. Anal. Appl.* **23** (1968) 421–427.